AFFINENESS OF DELIGNE-LUSZTIG VARIETIES FOR MINIMAL LENGTH ELEMENTS

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ABSTRACT. We prove that the Deligne-Lusztig varieties associated to elements of the Weyl group which are of minimal length in their twisted class are affine. Our proof differs from the proof of He and Orlik-Rapoport and it is inspired by the case of regular elements, which correspond to the varieties involved in Broué's conjectures.

1. INTRODUCTION

Let p be a prime number, let $\mathbb{F}$ denote an algebraic closure of the finite field with p elements and let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{F}$. We assume that $\mathbf{G}$ is endowed with an isogeny $F : \mathbf{G} \to \mathbf{G}$ such that F^δ is a Frobenius endomorphism with respect to some $\mathbb{F}_q$-structure on $\mathbf{G}$ (here, δ is a non-zero natural number, q is a power of p and $\mathbb{F}_q$ denotes the subfield of $\mathbb{F}$ with q elements).

We denote by $\boldsymbol{\mathcal{B}}$ the variety of Borel subgroups of $\mathbf{G}$ and by $\boldsymbol{\mathcal{B}} \times \boldsymbol{\mathcal{B}} = \coprod_{w \in W} \boldsymbol{\mathcal{O}}(w)$ the decomposition into orbits for the diagonal action of $\mathbf{G}$. Here, W is the Weyl group of $\mathbf{G}$, with set of simple reflections S corresponding to the orbits of dimension $1 + \dim \boldsymbol{\mathcal{B}}$, and the first and last projections define an isomorphism $\boldsymbol{\mathcal{O}}(w) \times_{\boldsymbol{\mathcal{B}}} \boldsymbol{\mathcal{O}}(w') \xrightarrow{\sim} \boldsymbol{\mathcal{O}}(ww')$ when $\ell(ww') = \ell(w) + \ell(w')$, where $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function on W associated to S.

Given $w \in W$, we define the *Deligne-Lusztig variety* [4, Definition 1.4] associated to w by

$$\mathbf{X}(w) = \mathbf{X}_{\mathbf{G}}(w) = \{\mathbf{B} \in \boldsymbol{\mathcal{B}} \mid (\mathbf{B}, F(\mathbf{B})) \in \boldsymbol{\mathcal{O}}(w)\}.$$

By studying a class of ample sheaves on $\mathbf{X}(w)$, Deligne and Lusztig proved that these varieties are affine when $q^{1/\delta}$ is larger than the Coxeter number of $\mathbf{G}$ [4, Theorem 9.7].

They proved more generally that the existence of coweights satisfying certain inequalities ensures that $\mathbf{X}(w)$ is affine. Recently, Orlik-Rapoport and He studied this question. Recall that $x, y \in W$ are *F-conjugate* if there exists $a \in W$ such that $y = a^{-1}xF(a)$. By a case-by-case analysis based on Deligne-Lusztig's criterion, they obtained the following result ([13, §5], [10, Theorem 1.3]):

Theorem A (Orlik-Rapoport, He). *If $w \in W$ is an element of minimal length in its F-conjugacy class then $\mathbf{X}(w)$ is affine.*

When w is a Coxeter element, the result is due to Lusztig [12, Corollary 2.8]. The aim of this note is to generalize, and give a more direct proof of Theorem A. As a consequence (and by applying a combinatorial result on elements of minimal length in their F-conjugacy class), we shall obtain a generalisation of Theorem A.

Before stating our results, we need some further notation. We denote by B^+ the *braid monoid* associated to (W, S). It is the monoid with presentation

$$B^+ = \langle (\underline{x})_{x \in W} \mid \forall\, x, x' \in W,\ \ell(xx') = \ell(x) + \ell(x') \Rightarrow \underline{xx'} = \underline{x}\,\underline{x'} \rangle.$$

The automorphism F of W extends to an automorphism of B^+ still denoted by F.

Given $I \subset S$, let W_I denote the subgroup of W generated by I and let w_I be the longest element of W_I (the element w_S will be denoted by w_0). The main result of this note is the following:

Theorem B. *Let I be an F-stable subset of S and let $w \in W_I$ be such that there exists a positive integer d and $a \in B^+$ with*

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w_I}a.$$

Then $\mathbf{X}(w)$ is affine.

The proof of Theorem B is by a general argument while our deduction of Theorem A relies on combinatorial results on finite Coxeter groups (see [7, Theorem 1.1], [6, §6] and [9, Theorem 7.5]) which are proved by a case-by-case analysis.

There is a case where our criterion can be applied easily. Indeed, if d is a regular number for (W, F) (in the sense of Springer) then by [2, Proposition 6.5], there exists a regular element $w \in W$ such that

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w_0}\underline{w_0}.$$

Therefore, by Theorem B, the variety $\mathbf{X}(w)$ is affine: this variety is of fundamental interest for the geometric version of Broué's abelian defect group conjecture for finite reductive groups [2, §5.B]. In particular, if $i \neq j$, this conjecture predicts that, as $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules, the cohomology groups $H_c^i(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)$ and $H_c^j(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)$ have no common irreducible constituents.

Finally, note that there exists elements satisfying the criterion of Theorem B but which do not satisfy Deligne-Lusztig's criterion. For instance, if W is of type B_5 (and F acts trivially on W), the element $w = s_1 t s_3 s_2 s_1 t s_1 s_4 s_3 s_2 s_1 t s_1 s_2 s_3$ does not satisfy Deligne-Lusztig's criterion (for $q = 2$) but satisfies $(\underline{w})^5 = (\underline{w_0})^3$ (here, $S = \{t, s_1, s_2, s_3, s_4\}$, $t s_1$ has order 4 and $s_i s_{i+1}$ has order 3 for $i = 1, 2, 3$). However, this element w is F-conjugate by cyclic shift (see Section 2 for the definition) to $s_4 w s_4 = s_1 t s_3 s_2 s_1 t s_1 s_2 s_3 s_4 s_3 s_2 s_1 t s_1$ which satisfies Deligne-Lusztig's criterion, so

the affineness of the variety $\mathbf{X}(w)$ can also be obtained from Deligne-Lusztig's criterion (see Proposition 2). These computations have been checked using `GAP3/CHEVIE` programs written by Jean Michel.

2. Preliminaries

Levi subgroup. Let us fix an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ and an F-stable maximal torus $\mathbf{T}$ of $\mathbf{B}_0$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}_0$. We identify $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ with W by requiring that $(\mathbf{B}_0, w\mathbf{B}_0 w^{-1}) \in \mathcal{O}(w)$.

Let I be an F-stable subset of S, let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{P}$, let $\mathbf{V}_I$ denote the unipotent radical of $\mathbf{P}_I$ and let $\mathbf{L}_I$ denote the unique Levi subgroup of $\mathbf{P}_I$ containing $\mathbf{T}$. Given $w \in W_I$, there is an isomorphism [11, Lemma 3]

$$\mathbf{X}_{\mathbf{G}}(w) \xrightarrow{\sim} \mathbf{G}^F/\mathbf{V}_I^F \times_{\mathbf{L}_I^F} \mathbf{X}_{\mathbf{L}_I}(w).$$

In particular,

(1) $\mathbf{X}_{\mathbf{G}}(w)$ *is affine if and only if* $\mathbf{X}_{\mathbf{L}_I}(w)$ *is affine.*

Cyclic shift. If w, $w' \in W$, we say that w and w' are *F-conjugate by cyclic shift* (and we write $w \xleftrightarrow{F} w'$) if there exists three sequences $(x_i)_{1 \leq i \leq n}$, $(y_i)_{1 \leq i \leq n}$ and $(w_i)_{1 \leq i \leq n+1}$ of elements of W such that

(1) $w_1 = w$ and $w_{n+1} = w'$;
(2) for all $i \in \{1, 2, \ldots, n\}$, $w_i = x_i y_i$, $w_{i+1} = y_i F(x_i)$ and $\ell(w_i) = \ell(w_{i+1}) = \ell(x_i) + \ell(y_i)$.

The relation $\xleftrightarrow{F}$ is an equivalence relation. Two elements which are F-conjugate by cyclic shift have the same length.

Proposition 2. *If $w \xleftrightarrow{F} w'$, then $\mathbf{X}(w)$ is affine if and only if $\mathbf{X}(w')$ is affine.*

Proof - By induction, we may assume that there exists x and y in W such that $w = xy$, $w' = yF(x)$ and $\ell(w) = \ell(w') = \ell(x) + \ell(y)$. The result follows from the existence of a purely inseparable morphism $\mathbf{X}(w) \to \mathbf{X}(w')$ [4, Page 108]. ∎

3. Proof of Theorem B

Let I be an F-stable subset of S, let $w \in W_I$ and assume that there exists $a \in B^+$ and a positive integer d such that

$$\underline{w} F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_I a.$$

The aim of this section is to prove that $\mathbf{X}(w)$ is affine (Theorem B). By (1), we may (and we will) assume that $I = S$.

Sequences of elements of W. Given $(x_1, \ldots, x_r)$ a sequence of elements of W, we set

$$\mathcal{O}(x_1, \ldots, x_r) = \mathcal{O}(x_1) \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{O}(x_r).$$

If $(y_1, \ldots, y_s)$ is a sequence of elements of W such that $\underline{x}_1 \cdots \underline{x}_r = \underline{y}_1 \cdots \underline{y}_s$ in B^+, then $\mathcal{O}(x_1, \ldots, x_r) \simeq \mathcal{O}(y_1, \ldots, y_s)$ (the varieties are actually canonically isomorphic [3, Application 2]). For a general treatment of these varieties $\mathcal{O}(x_1, \ldots, x_r)$ (and the corresponding Deligne-Lusztig varieties), the reader may refer to [5].

Proposition 3. *If there exists $v \in B^+$ such that $\underline{x}_1 \cdots \underline{x}_r = \underline{w}_0 v$, then the variety $\mathcal{O}(x_1, \ldots, x_r)$ is affine.*

PROOF - Let $v_1, \ldots, v_n \in W$ be such that $\underline{v}_1 \cdots \underline{v}_n = v$. We have $\mathcal{O}(x_1, \ldots, x_r) \simeq \mathcal{O}(w_0, v_1, \ldots, v_n)$, so it remains to prove that $\mathcal{O}(w_0, v_1, \ldots, v_n)$ is affine.

For each $x \in W$, we fix a representative $\dot{x}$ of x in $N_{\mathbf{G}}(\mathbf{T})$. We set

$$\tilde{\mathcal{O}}(x_1, \ldots, x_r) = \{(g_0 \mathbf{U}, g_1 \mathbf{U}, \ldots, g_r \mathbf{U}) \in (\mathbf{G}/\mathbf{U})^{r+1} \mid \forall \, 1 \le i \le r, \ g_{i-1}^{-1} g_i \in \mathbf{U} \dot{x}_i \mathbf{U}\}.$$

The group $\mathbf{T}$ acts on the right on $\tilde{\mathcal{O}}(x_1, \ldots, x_r)$ as follows:

$$(g_0 \mathbf{U}, g_1 \mathbf{U}, \ldots, g_r \mathbf{U}) * t = (g_0 t \mathbf{U}, g_1^{x_1^{-1}} t \mathbf{U}, \ldots, g_r^{x_r^{-1} \cdots x_1^{-1}} t \mathbf{U}).$$

The canonical map

$$\begin{array}{ccc} \tilde{\mathcal{O}}(x_1, \ldots, x_r) & \longrightarrow & \mathcal{O}(x_1, \ldots, x_r) \\ (g_0 \mathbf{U}, g_1 \mathbf{U}, \ldots, g_r \mathbf{U}) & \longmapsto & (g_0 \mathbf{B}_0 g_0^{-1}, g_1 \mathbf{B}_0 g_1^{-1}, \ldots, g_r \mathbf{B}_0 g_r^{-1}) \end{array}$$

identifies $\mathcal{O}(x_1, \ldots, x_r)$ with the quotient of $\tilde{\mathcal{O}}(x_1, \ldots, x_r)$ by $\mathbf{T}$: indeed, both varieties are smooth (hence normal), the above map is smooth (hence separable) and it is easily checked that its fibers are precisely the $\mathbf{T}$-orbits. Since $\mathbf{T}$ acts freely on $\tilde{\mathcal{O}}(x_1, \ldots, x_r)$, and since the quotient of an affine variety by a free action of a torus is affine, [1, Corollary 8.21], the result will follow if we are able to prove that $\tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n)$ is affine. Therefore, it is sufficient to show that the map

$$\varphi : \mathbf{G} \times \prod_{i=1}^{n} (\mathbf{U} \dot{v}_i \cap \dot{v}_i \mathbf{U}^-) \longrightarrow \tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n)$$
$$(g; h_1, \ldots, h_n) \longmapsto (g\mathbf{U}, g\dot{w}_0 \mathbf{U}, g\dot{w}_0 h_1 \mathbf{U}, g\dot{w}_0 h_1 h_2 \mathbf{U}, \ldots, g\dot{w}_0 h_1 \cdots h_n \mathbf{U})$$

is an isomorphism of varieties (here, $\mathbf{U}^- = {}^{w_0}\mathbf{U}$).

In order to prove that φ is an isomorphism, we shall construct its inverse. For this, we shall need some notation. First, the map $\mathbf{U} \times \mathbf{U} \to \mathbf{U}\dot{w}_0\mathbf{U}$, $(x, y) \mapsto x\dot{w}_0 y$ is an isomorphism of varieties: we shall denote by $\mathbf{U}\dot{w}_0\mathbf{U} \to \mathbf{U} \times \mathbf{U}$, $g \mapsto (\eta(g), \eta'(g))$ its inverse. Also, the map $\mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^- \times \mathbf{U} \to \mathbf{U}\dot{v}_i\mathbf{U}$, $(x, y) \mapsto xy$ is an isomorphism of varieties $(i = 1, 2, \ldots, n)$, and we shall denote by $\eta_i : \mathbf{U}\dot{v}_i\mathbf{U} \to \mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^-$ the composition ot its inverse with the first projection. Note that, if $g \in \mathbf{U}\dot{w}_0\mathbf{U}$, $h \in \mathbf{U}\dot{v}_i\mathbf{U}$ and $u, v \in \mathbf{U}$, then

$$(*) \quad \eta(ugv) = u\eta(g), \quad \eta(g)\dot{w}_0\mathbf{U} = g\mathbf{U}, \quad \eta_i(hv) = \eta_i(h) \quad \text{and} \quad \eta_i(h)\mathbf{U} = h\mathbf{U}.$$

Now, if $x = (g\mathbf{U}, g_0\mathbf{U}, g_1\mathbf{U}, \ldots, g_n\mathbf{U}) \in \tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n)$, we set

$$\begin{aligned}
\psi(x) &= g\eta(g^{-1}g_0), \\
\psi_0(x) &= \psi(x)\dot{w}_0, \\
\psi_i(x) &= \eta_i\big((\psi_0(x)\psi_1(x)\cdots\psi_{i-1}(x))^{-1}g_i\big),
\end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. By $(*)$, the maps ψ and ψ_j are well-defined morphisms of varieties and it is easily checked that the morphism of varieties

$$\begin{aligned}
\tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n) &\longrightarrow \mathbf{G} \times \prod_{i=1}^{n}(\mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^-) \\
x &\longmapsto (\psi(x); \psi_1(x), \ldots, \psi_n(x))
\end{aligned}$$

is well-defined and is an inverse of φ. ∎

Introducing Frobenius. The morphism

$$\mathbf{X}(w) \to \mathcal{B}^d, \ \mathbf{B} \mapsto (\mathbf{B}, F(\mathbf{B}), \ldots, F^{d-1}(\mathbf{B}))$$

indentifies $\mathbf{X}(w)$ with the closed subvariety $\Delta_d \cap \mathcal{O}(w, F(w), \ldots, F^{d-1}(w))$, where $\Delta_d = \{(\mathbf{B}, F(\mathbf{B}), \ldots, F^{d-1}(\mathbf{B})) \mid \mathbf{B} \in \mathcal{B}\}$ is a closed subvariety of $\mathcal{B}^d$. By Proposition 3, the variety $\mathcal{O}(w, F(w), \ldots, F^{d-1}(w))$ is affine, hence $\mathbf{X}(w)$ is affine as well. The proof of Theorem B is complete.

4. Proof of Theorem A

Let C be an F-conjugacy class in W and $C_{\min}$ its subset of elements of minimal length. Let d be the smallest positive integer k such that $wF(w)\ldots F^{k-1}(w) = 1$ and F^k acts as the identity on W for $w \in C_{\min}$. Following [7, Theorem 1.1] (in the split case) and [6, Definition 5.3] (in the general case), we say that an element $w \in C_{\min}$ is *good* if there exists a sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r$ of subsets of S such that

$$(*) \qquad\qquad \underline{w}F(\underline{w})\ldots F^{d-1}(\underline{w}) = \underline{w}_{I_1}^2\underline{w}_{I_2}^2\cdots\underline{w}_{I_r}^2$$

in B^+.

Proposition 4. *If w is a good element of $C_{\min}$, then $\mathbf{X}(w)$ is affine.*

PROOF - By Theorem B, it remains to show that the subset I_1 of the identity $(*)$ is F-stable. Let I be the set of simple reflections occuring in a reduced expression of w (note that I does not depend on the choice of the reduced expression [8, Corollary 1.2.3]). Then the set of $s \in S$ such that $\underline{s}$ occurs in a reduced expression of $\underline{w}F(\underline{w})\ldots F^{d-1}(\underline{w})$ is equal to $I \cup F(I) \cup \cdots \cup F^{d-1}(I)$ (by looking at the left-hand side of $(*)$) and is also equal to I_1 (by looking at the right-hand side). Since F^d acts as the identity on W, we get that I_1 is F-stable. ■

Let us now come back to the proof of Theorem A. Let $w \in C_{\min}$. Let I be the minimal F-stable subset of S such that $w \in W_I$. By (1), we may assume that $I = S$. Now, if $w' \in C_{\min}$, then $w \xleftrightarrow{F} w'$ (see [8, Theorem 3.2.7] for the split case, [6, §6] for twisted exceptional groups and [9, Theorem 7.5] for twisted classical groups), so $\mathbf{X}(w)$ is affine if and only if $\mathbf{X}(w')$ is affine by Proposition 2. Therefore, the result follows from Proposition 4 and the next Theorem:

Theorem 6 (Geck-Michel, Geck-Kim-Pfeiffer, He). *There exists a good element in $C_{\min}$.*

PROOF - By standard arguments (see [6, §5.5]), we may assume that W is irreducible. If F acts trivially on W, the Theorem is [7, Theorem 1.1]. If F does not act trivially and W is not of type A, this is [6, §5.5]. When W is of type A and F acts non trivially on W, this follows from [9, Corollary 7.25]. ■

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